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Existence of Chaos in Control Systems with Delayed Feedback

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1. INTRODUCTION

In this paper we prove the existence of very complicated solutions to the difference–differential equation

$$\dot{x}(t) = f(x(t-1)) - \alpha x(t). \tag{1.1}$$

Here $f\colon \mathbb{R} \to \mathbb{R}$ denotes some (nonlinear) function, α a positive constant, $t \in \mathbb{R}$ is the independent variable, $x \in \mathbb{R}$ the dependent variable, and $\dot{x} = dx/dt$. The investigation is motivated (i) by the applications Eq. (1.1) has found in several areas of applied sciences and (ii) by the observation of intensely structured periodic and aperiodic oscillations in numerical simulations. For a certain class of nonlinearities f we show that these structures are not numerical artefacts but properties of the solution manifold of Eq. (1.1). In the applications Eq. (1.1) has been used to explain normal and pathological behavior of control systems in the physiology of blood cell production and respiration [7, 8, 10, 11] and periodic [2] or irregular [5] activity in neural networks. Computer simulations to Eq. (1.1) may be found in [3]. The mathematical analysis of this equation so far has been restricted to prove the existence of slowly oscillating, simple periodic solutions [1, 4, 6]. A periodic solution is called slowly oscillating if successive

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inflection points are spaced apart by at least the length of the delay (here normalized to 1). It is called simple if there is exactly one maximum within one smallest period.

Equation (1.1) is in some sense a continuous version of the difference equation

$$z_{n+1} = g(z_n), \tag{1.2}$$

$n \in \mathbb{N}$, $g\colon \mathbb{R} \to \mathbb{R}$ continuous, $z_n \in \mathbb{R}$. This may be seen most easily when (1.1) is written as $\varepsilon\dot{x}(t) + x(t) = g(x(t-1))$ and $\varepsilon \to 0$. In their paper [9] Li and Yorke proved that Eq. (1.2) displays chaos if there is some $r \in \mathbb{R}$ such that

$$g^3(r) \leqslant r < g(r) < g^2(r). \tag{1.3}$$

They defined chaos to be present if Eq. (1.2) has infinitely many periodic solutions with different periods and if additionally there is an uncountable set $S \subset \mathbb{R}$ such that

$$\text{(LY)} \quad \left\{ \begin{array}{ll} \left. \begin{array}{l} \limsup\limits_{n\to\infty} |g^n(p) - g^n(q)| > 0 \\ \liminf\limits_{n\to\infty} |g^n(p) - g^n(q)| = 0 \end{array} \right\} & \text{if} \quad p, q \in S,\ p \neq q \\ \text{and} & \\ \limsup\limits_{n\to\infty} |g^n(p) - g^n(\tilde{q})| > 0 & \text{if} \quad p \in S \text{ and if } (g^n(\tilde{q}))_{n\in\mathbb{N}} \\ & \qquad \text{is periodic.} \end{array} \right.$$

It is precisely this type of chaos we shall prove to exist for the continuous Eq. (1.1), too: For suitable f there is a subset of initial conditions such that the corresponding solutions $x(t)$, $t > 0$, oscillate around some level a. The values t, where $x(t) = a$, form a sequence $(a_n)_{n=1,2,\ldots}$, $0 < a_n < a_{n+1}$ such that $(m_n)_{n=1,2,\ldots}$ with $m_n = a_{2n} - a_{2n-1}$ obeys a difference equation exhibiting chaos in the sense of Li and Yorke (compare Theorem 5).

However, our result is not so much related to criterion (1.3) in its original form but to the generalization of Marotto [12] who proved chaos for maps $g\colon \mathbb{R}^n \to \mathbb{R}^n$ with fixed points $\bar{z}$ which are snap-back repellers. This condition includes the existence of a homoclinic solution $(g^n(\tilde{z}))_{n\in\mathbb{Z}}$ such that $\tilde{z} \neq \bar{z}$, $g^n(\tilde{z}) \to \bar{z}$ as $n \to -\infty$, $g^n(\tilde{z}) = \bar{z}$ for $n \geqslant n_0$, with some $n_0 \in \mathbb{N}$.

Indeed we shall solve the problem for the functional differential equation (1.1) by constructing a solution (later on called $y_{\bar{a}}$) which is homoclinic to an unstable periodic orbit, and stays on the periodic orbit for t in some unbounded interval in $\mathbb{R}^+$. We obtain a Poincaré map T on a one-dimensional subset in the state space C of continuous functions on $[-1, 0]$, with homoclinic solution $(T^n(\varphi))_{n \in \mathbb{Z}}$, φ the initial value $y_{\bar{a}} | [-1, 0]$. Technically, this is made possible by the choice of the nonlinearity f in Eq. (1.1) close to a step function but nevertheless smooth.

The criterion to be used then for T (in terms of a coordinate) is an easy consequence of arguments due to Li, Yorke and Marotto, but for maps which do not let their domain invariant:

THEOREM 1. *Let I, U be disjoint compact intervals of finite length. Let a map $g: I \cup U \to \mathbb{R}$ be given with $g(I) \supset I \cup U$. Assume $|dg/dx| > 1$ on I, and that $g | U$ is a homeomorphism onto a neighborhood of the fixed point of g in I.*

Then there exists $n_0 \in \mathbb{N}$ such that the maximal positively invariant set $M \subset I \cup U$ contains periodic orbits of every length $n \geqslant n_0$, and an uncountable set $S \subset M$ without periodic orbits which satisfies (LY).

For a proof, see [13]. The method has already been successful for the somewhat simpler equation

$$\dot{x}(t) = f(x(t-1))$$

investigated by one of the authors [13]. However, the term $\alpha x(t)$ depending on presence generates many peculiarities and difficulties, all the more the paper demonstrates the broader applicability of the technique and that chaos is not bound to special equations. Moreover the treatment of Eq. (1.1) appears urgent in view of its relevance in applied sciences.

2. THE DIFFERENTIAL EQUATION

Throughout the paper α, a, b, c, d, $\delta \in \mathbb{R}$ are constants satisfying

$$0 < \alpha, \qquad 0 < \delta < a < 1, \qquad 1 + \delta \leqslant b < c/\alpha, \qquad d \leqslant 0.$$

Let f: $\mathbb{R} \to \mathbb{R}$ be a continuous function having the following properties:

$$\begin{aligned} f(x) &= 0 && \text{if} \quad x < a - \delta \quad \text{or} \quad 1 + \delta < x < b \\ &= c && \text{if} \quad a < x < 1 \\ &= d && \text{if} \quad b + \delta < x. \end{aligned}$$

On each of the intervals $(a - \delta, a)$, $(1, 1 + \delta)$, $(b, b + \delta)$ the function f is assumed to satisfy

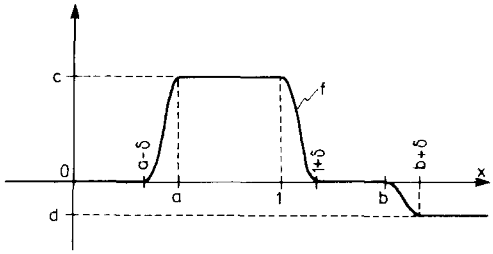

FIGURE 1

$$\begin{aligned} f(x) &= c \cdot g_1((x-a+\delta)/\delta) && \text{if} \quad a-\delta < x < a \\ &= c \cdot g_2((1+\delta-x)/\delta) && \text{if} \quad 1 < x < 1+\delta \\ &= d \cdot g_3((x-b)/\delta) && \text{if} \quad b < x < b+\delta, \end{aligned}$$

where g_i: $[0, 1] \to [0, 1]$ are continuous, monotone functions obeying $g_i(0) = 0$, $g_i(1) = 1$ for $i = 1, 2, 3$.

We study solutions of the differential-delay equation

$$\dot{x}(t) = f(x(t-1)) - \alpha x(t). \tag{2.1}$$

An initial condition to (2.1) is a function $\varphi \in C = C([-1, 0])$, the space of continuous functions ψ: $[-1, 0] \to \mathbb{R}$ with supremum-norm.

The formal integration of Eq. (2.1)

$$x(t) = x(t_0)\, e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-s)} f(x(s-1))\, ds \tag{2.2}$$

$(t > t_0)$ shows that any initial condition φ determines a unique solution $x = x_\varphi$: $[-1, \infty) \to \mathbb{R}$ such that

$$x|_{[-1,0]} = \varphi, \qquad x \text{ obeys (2.1) for all } t > 0.$$

It is easy to see that $d/\alpha \leqslant \varphi \leqslant c/\alpha$ implies $d/\alpha \leqslant x(t) \leqslant c/\alpha$ for all $t > 0$. With

$$\gamma = c/\alpha$$

Eq. (2.2) implies

$$
\begin{aligned}
x(t) &= x(t_0)\, e^{-\alpha(t-t_0)} && \text{if} \quad x(s-1) \leqslant a-\delta \\
&= \gamma + (x(t_0)-\gamma)\, e^{-\alpha(t-t_0)} && \text{if} \quad a \leqslant x(s-1) \leqslant 1 \\
&= x(t_0)\, e^{-\alpha(t-t_0)} && \text{if} \quad 1+\delta \leqslant x(s-1) \leqslant b \\
&= \frac{d}{\alpha} + \left(x(t_0) - \frac{d}{\alpha}\right) e^{-\alpha(t-t_0)} && \text{if} \quad b+\delta \leqslant x(s+1)
\end{aligned}
$$

$$\text{for all} \quad s \in [t_0, t]. \tag{2.3}$$

Let the numbers $\eta_1 = \eta_1(\delta) > 0$ and $\eta_2 = \eta_2(\delta) > 0$ be defined by

$$a - \delta = a \cdot \exp(-\alpha \eta_1), \tag{2.4}$$

$$a = \gamma + (a - \delta - \gamma) \exp(-\alpha \eta_2). \tag{2.5}$$

Observe that

$$\eta_1 \to 0 \qquad \text{and} \qquad \eta_2 \to 0 \qquad \text{as} \quad \delta \to 0. \tag{2.6}$$

The investigation is restricted to the following set $D \subset C$ of initial conditions.

DEFINITION 1. *Let $\eta_1 + \eta_2 < 1$. $\varphi \in D$ if and only if the following conditions hold*

(i) $\varphi(0) = a$,

(ii) $\varphi(t) = \gamma + (a - \gamma) \exp(-\alpha t)$ *if* $t \in [-\eta_2, 0]$,

(iii) *there is a number* $v \in [\eta_1 + \eta_2, 1]$ *such that*

$$\varphi(t) = a \exp(-\alpha[t - (-v)]) \qquad \textit{if} \quad t \in [-v, -v + \eta_1],$$

(iv) $a \leqslant \varphi(t) \leqslant 1$ *if* $t \in [-1, -v]$

(v) $\varphi(t) \leqslant a - \delta$ *if* $t \in [-v + \eta_1, -\eta_2]$.

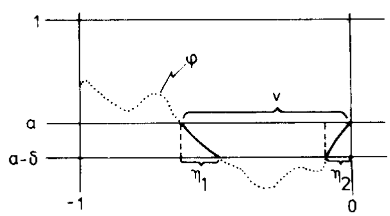

FIGURE 2

The properties of D induce a continuous map

$$V\colon D \to [\eta_1 + \eta_2, 1], \qquad V(\varphi) = v.$$

PROPOSITION 1. *Let x, y be the solutions corresponding to $\varphi, \psi \in D$, respectively. If $V(\varphi) = V(\psi)$, then $x(t) = y(t)$ for all $t \geqslant 0$.*

Proof. Formulas (2.2) and (2.3) imply

$$x(t) = \gamma + (a - \gamma) \exp(-\alpha t) \quad \text{for} \quad t \in [0, 1 - v], \tag{2.7}$$

$$x(t) = x(1 - v) \exp(-\alpha[t - (1 - v)]) + E_1(t) \quad \text{for} \quad t \in [1 - v, 1 - v + \eta_1], \tag{2.8}$$

where

$$E_1(t) = \int_{1-v}^{t} e^{-\alpha(t-s)} f(a \exp(-\alpha[s - (-v) - 1])) \, ds, \tag{2.9}$$

$$x(t) = x(1 - v + \eta_1) \exp(-\alpha[t - (1 - v + \eta_1)]) \quad \text{for} \quad t \in [1 - v + \eta_1, 1 - \eta_2], \tag{2.10}$$

$$x(t) = x(1 - \eta_2) \exp(-\alpha[t - (1 - \eta_2)]) + E_2(t) \quad \text{for} \quad t \in [1 - \eta_2, 1], \tag{2.11}$$

where

$$E_2(t) = \int_{1-\eta_2}^{t} e^{-\alpha(t-s)} f(\gamma + (a - \gamma) e^{-\alpha(s-1)}) \, ds. \tag{2.12}$$

The same formulas hold for y. Therefore, $x(t) = y(t)$ for $t \in [0, 1]$, hence $x(t) = y(t)$ for all $t \geqslant 0$. Q.E.D.

3. UNSTABLE PERIODIC SOLUTION AND POINCARÉ MAP

Throughout this section we assume

Condition 1. Defining $\gamma = c/\alpha$ and $\rho = (\gamma - a)/(\gamma - 1)$ let

$$1 + \gamma(1 - a^{-1}) < \rho e^{-\alpha} \leqslant a, \tag{3.1}$$

and let δ be sufficiently small.

The latter condition means that there is $\delta_0 > 0$ such that all subsequent assertions hold for all $\delta \in (0, \delta_0)$.

LEMMA 3.1. *There are numbers $v_1 = v_1(\delta)$, $v_2 = v_2(\delta)$, $\eta_1 + \eta_2 \leqslant v_1 < v_2 \leqslant 1$, such that $\varphi \in D$, $v_1 < V(\varphi) = v < v_2$ implies*

$$a \leqslant x(t) \leqslant 1 \quad \textit{for} \quad t \in [0, 1 - v + \eta_1]. \tag{3.2}$$

Additionally

$$\lim_{\delta\to 0} v_1 = 1 - \frac{1}{\alpha}\log\rho, \qquad \lim_{\delta\to 0} v_2 = 1. \tag{3.3}$$

Proof. Let $\varphi \in D$. According to (2.7) the solution $x = x_\varphi$ satisfies (3.2) for $t \in [0, 1-v]$ iff

$$x(1-v) = \gamma + (a-\gamma)\exp(-\alpha[1-v]) \leqslant 1, \tag{3.4}$$

i.e., iff

$$v \geqslant 1 - \frac{1}{\alpha}\log\rho.$$

For $t \in [1-v, 1-v+\eta_1]$ Eqs. (2.4) and (2.8) imply

$$x(1-v)\cdot(a-\delta)/a \leqslant x(t) \leqslant x(1-v) + E_1(t) \leqslant x(1-v) + \eta_1 c. \tag{3.5}$$

If v increases from $1-(1/\alpha)\log\rho$ to 1, then $x(1-v)$ decreases from 1 to a. Because of (2.6) and the second inequality in (3.1) there are uniquely determined numbers v_1, v_2 such that

$$\begin{aligned} x(1-v_2)\cdot\frac{a-\delta}{a} &= a, \\ x(1-v_1) + \eta_1\cdot c &= \frac{a-\delta-\eta_2 c}{a}\cdot\frac{\gamma-a}{\gamma-a+\delta} < 1. \end{aligned} \tag{3.6}$$

These numbers satisfy (3.2) and (3.3). Q.E.D.

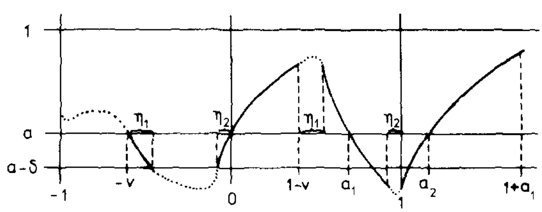

FIGURE 3

LEMMA 3.2. *For all* $\varphi \in V^{-1}([v_1, v_2])$

$$x(t) \leqslant a-\delta \qquad \textit{if} \quad t \in [1-\eta_2, 1]. \tag{3.7}$$

Moreover

$$\lim_{\delta\to 0} v_1 - \frac{1}{\alpha}\log a.$$

Proof. Let $\varphi \in V^{-1}([v_1, v_2])$. Because of (2.8) and (2.6)

$$x(1-v+\eta_1) = x(1-v)\frac{a-\delta}{a} + \kappa_1, \tag{3.8}$$

the constant

$$\kappa_1 = \kappa_1(\delta) = \frac{a-\delta}{a}\int_0^{\eta_1} e^{\alpha s} f(ae^{-\alpha s})\, ds$$

satisfying

$$\kappa_1 \leqslant \frac{a-\delta}{a}\cdot c\cdot \eta_1 \qquad \text{and} \qquad \lim_{\delta\to 0}\kappa_1 = 0. \tag{3.9}$$

For $t \in [1-v+\eta_1, 1-\eta_2]$ the solution x obeys (2.10). Hence

$$x(1-\eta_2) = x(1-v+\eta_1)\exp(-\alpha v)\frac{a}{a-\delta}\cdot\frac{\gamma-a+\delta}{\gamma-a}. \tag{3.10}$$

According to (2.11)

$$x(t) \leqslant x(1-\eta_2) + \eta_2\cdot c \qquad \text{for} \quad t\in[1-\eta_2, 1]. \tag{3.11}$$

$x(1-\eta_2)$ is a decreasing function of v (since $x(1-v)$ is decreasing). Hence with (3.10), (3.8) and (3.9)

$$\begin{aligned} x(1-\eta_2) + \eta_2\cdot c \leqslant (x(1-v_1) + c\eta_1)\frac{a-\delta}{a}\cdot e^{-\alpha v_1} \\ \cdot\frac{a}{a-\delta}\cdot\frac{\gamma-a+\delta}{\gamma-a} + \eta_2 c. \end{aligned} \tag{3.12}$$

Inequality (3.4) and Condition 1 ($\rho e^{-\alpha} \leqslant 1$) imply

$$e^{-\alpha v_1} \leqslant a.$$

Therefore

$$x(1-\eta_2) + \eta_2\cdot c \leqslant (x(1-v_1)+c\eta_1)\, a\,\frac{\gamma-a+\delta}{\gamma-a} + \eta_2 c = a-\delta, \tag{3.13}$$

the equality following from the choice (3.6) of v_1. The lemma results from (3.11) and (3.13). Q.E.D.

By the preceding lemmas we know that every solution x corresponding to an initial condition $\varphi \in V^{-1}([v_1, v_2])$ has the following property:

There is a time $t = a_1 = a_1(v)$, $0 \leqslant a_1 \leqslant 1$, such that

$$x(a_1) = a \qquad \text{and} \qquad x(t) = a e^{-\alpha(t-a_1)} \qquad \text{for} \quad t \in [a_1, a_1 + \eta_1].$$

Since

$$1 - v + \eta_1 \leqslant a_1 \leqslant 1 - \eta_2, \tag{3.14}$$

Eq. (2.10) implies

$$a_1 = 1 - v + \eta_1 + \frac{1}{\alpha} \log \frac{x(1 - v + \eta_1)}{a}. \tag{3.15}$$

a_1 is a decreasing function of v.

LEMMA 3.3. *There is a number $v_3 = v_3(\delta)$, $v_1 < v_3 \leqslant v_2$, such that for all $\varphi \in V^{-1}([v_1, v_3))$*

$$x(a_1 + 1) > a \tag{3.16}$$

and

$$x(a_1 + 1) = a \qquad \textit{for} \quad \varphi \in V^{-1}(v_3). \tag{3.17}$$

Proof. Let $\varphi \in V^{-1}([v_1, v_2])$. Since

$$x(t) \in [a, 1] \qquad \text{for} \quad t \in [0, a_1],$$

$$x(t) = \gamma + (x(1) - \gamma)\, e^{-\alpha(t-1)} \qquad \text{for} \quad t \in [1, 1 + a_1]. \tag{3.18}$$

Because of (3.10), (2.11), and (2.5)

$$x(1) = x(1 - \eta_2) \frac{\gamma - a + \delta}{\gamma - a} + \kappa_2, \tag{3.19}$$

$$\kappa_2 = \int_{-\eta_2}^{0} e^{\alpha s} f(\gamma + (a - \gamma)\, e^{-\alpha s})\, ds$$

satisfying $\lim_{\delta \to 0} \kappa_2 = 0$. As a function of v

$$x(1 + a_1) = \gamma - (\gamma - x(1))\, e^{-\alpha a_1}$$

is strictly decreasing. Therefore, a number v_3 wirh property (3.16) exists iff

$$x(1+a_1)\,|_{v_1} > a, \tag{3.20}$$

where the left-hand side denoted the value $x(1+a_1)$ of any solution $x = x_\varphi$ with $\varphi \in V^{-1}(v_1)$, and $a_1 = a_1(v_1)$.

Since $v_1 = v_1(\delta)$ and since $\lim_{\delta \to 0} x(1+a_1)|_{v_1}$ exists, inequality (3.20) holds for sufficiently small $\delta > 0$ if

$$\lim_{\delta \to 0} x(1+a_1)|_{v_1} > a. \tag{3.21}$$

Because of (3.8), (3.6), and

$$a = x(1 - v_1 + \eta_1)\, e^{-\alpha(a_1 - [1 - v + \eta_1])}$$

we have

$$\lim_{\delta \to 0} e^{-\alpha a_1} e^{\alpha(1 - v_1)} = a.$$

Hence

$$\begin{aligned}\lim_{\delta \to 0} x(1+a_1)|_{v_1} &= \lim_{\delta \to 0} \left[\gamma - (\gamma - x(1))\, a e^{-\alpha(1 - v_1)}\right] \\ &= \lim_{\delta \to 0} \left[\gamma - (\gamma - e^{-\alpha v_1})\, a e^{-\alpha(1 - v_1)}\right] \\ &= \lim_{\delta \to 0} \left[\gamma - a e^{-\alpha}(\gamma e^{\alpha v_1} - 1)\right].\end{aligned}$$

Therefore by use of (3.3) condition (3.21) is equivalent to

$$\gamma/a > 1 + \gamma \frac{\gamma - 1}{\gamma - a} - e^{-\alpha},$$

which follows from the first part of Condition 1. Hence (3.16) is proved. A number $v_3 = v_3(\delta) \in (v_1, v_2)$ satisfying both (3.16) and (3.17) exists provided

$$\lim_{\delta \to 0} x(a_1 + 1)|_{v_2} < a.$$

Indeed this limit euals $a \cdot \exp(-\alpha)$, as $\lim_{\delta \to 0} v_2 = 1$. Q.E.D.

Let

$$I = [v_1, v_3]. \tag{3.22}$$

It follows from (3.7) and (3.16)–(3.18) that for each $\varphi \in V^{-1}(I)$ there is a unique time $t = a_2 \in (1, 1 + a_1]$ such that $x(a_2) = a$. Obviously

$$a_2 = 1 + \frac{1}{\alpha} \log \frac{\gamma - x(1)}{\gamma - a}. \tag{3.23}$$

We are now able to define a "Poincaré map"

$$T\colon V^{-1}(I) \to D$$

in the following way:

If x_φ denotes the solution corresponding to $\varphi \in V^{-1}(I)$, define

$$\psi(t) = T(\varphi)(t) = x_\varphi(a_2 + t), \qquad t \in [-1, 0]. \tag{3.24}$$

Because of the Lemmas 3.1 and 3.2 and formulas (2.10), (3.18) indeed $T\varphi \in D$. Closely related to T is the function $F\colon I \to (0, 1]$, given by

$$F(v) = V(T(\varphi)) = a_2 - a_1 \tag{3.25}$$

where $a_1 = a_1(v)$, $a_2 = a_2(v)$ are the first and second positive times, respectively, when the solution corresponding to φ, $V(\varphi) = v$, has the value a.

LEMMA 3.4. *The function F satisfies*

(i) $F(v_3) = 1$,

$$\begin{aligned} \text{(ii)}\quad \lim_{\delta \to 0} F(v_1) &= 1 + \frac{1}{\alpha} \log(a(\gamma - \rho e^{-\alpha})(\gamma - 1)/(\gamma - a)^2) \\ &= \lim_{\delta \to 0} v_1 + \frac{1}{\alpha} \log(a(\gamma - \rho e^{-\alpha})/(\gamma - a)), \end{aligned} \tag{3.26}$$

(iii) $\dfrac{\partial F}{\partial v} > 1$ *if δ is sufficiently small.*

Proof. (i) is trivial since $a_2(v_3) = 1 + a_1(v_3)$. To prove (ii) observe that successive application of (3.15), (2.6), (3.3), (3.8), (3.6) yields

$$\lim_{\delta \to 0} a_1(v_1) = \frac{1}{\alpha} \log \frac{1}{a} \frac{\gamma - a}{\gamma - 1}.$$

Hence (ii) follows from (3.25), (3.23) and the fact that $\lim_{\delta \to 0} x(1)|_{v_1} = (\gamma - a)\, e^{-\alpha}/(\gamma - 1)$ (see (3.3), (3.6), and (3.10)).

$$F(v) = a_2 - a_1 = \frac{1}{\alpha} \log \frac{\gamma - x(1)}{\gamma - a} + v - \eta_1 - \frac{1}{\alpha} \log \frac{x(1 - v + \eta_1)}{a}. \tag{3.27}$$

Previous calculations ((2.11), (3.10), (3.8), (3.4)) show that $x(1)$ and $x(1-v-\eta_1)$ are differentiable and strictly decreasing with respect to v, which implies (iii). Q.E.D.

THEOREM 2. *Let Condition 1 hold. Then the function F has a unique fixed point v_p. The solution x_p corresponding to the initial condition $\varphi_p = T\varphi$ with $\varphi \in D$, $V(\varphi) = v_p$, is periodic with minimal period $a_2(v_p)$, $1 < a_2(v_p) < 2$. The periodic solution x_p is unstable.*

Proof. It follows from the first inequality of Condition 1 that

$$\lim_{\delta \to 0} F(v_1) < \lim_{\delta \to 0} v_1.$$

Lemma 3.4 implies the unique existence of v_p, $F(v_p) = v_p$. Hence the existence of the unstable periodic solution x_p follows from $T\psi \in D$ for $\psi \in V^{-1}(I)$, Proposition 1 and $\partial F/\partial v > 1$. Q.E.D.

We are now able to restrict T to a one-dimensional subset $X \subset C([-1, 0])$, which is locally invariant.

LEMMA 3.5. *Let $X = T(V^{-1}(I))$. Then $V|_X$ is a homeomorphism from X to $[Fv_1, 1] = F(I)$, and $T(X \cap V^{-1}(I)) = X$.*

Proof. $V(T(V^{-1}(I)) = F(V(V^{-1}(I))) = F(I)$. $V|_X$ is injective as a consequence of Proposition 1 and of Lemma 3.4(iii); V is continuous by definition. Q.E.D.

4. FURTHER, MORE COMPLEX PERIODIC SOLUTIONS

For simplicity the investigation is restricted to the following set of parameters:

Condition 2. Let δ be positive and sufficiently small,

$$\alpha \geqslant \log 4, \qquad a = e^{-\alpha/2}, \qquad c = (1+a)\,\alpha, \qquad b = 1 + a - a^{5/2}. \tag{4.1}$$

Condition 2 is assumed to hold throughout Sections 4–6. All conclusions of the preceding section hold since Condition 2 implies Condition 1. Note that

$$\gamma = \frac{c}{\alpha} = 1 + a, \qquad \frac{\gamma - 1}{\gamma - a} = \frac{1}{\rho} = a, \tag{4.2}$$

$$\lim_{\delta \to 0} v_1 = \tfrac{1}{2}, \qquad \lim_{\delta \to 0} F(v_1) = 0. \tag{4.3}$$

LEMMA 4.1. *Let* $w = w(\delta) \in (F(v_1), F(v_1) + \delta) \subset \mathbb{R} \backslash I$. *Let* $\varphi \in X$, $V(\varphi) = w$. *Then the corresponding solution* $y = x_\varphi$ *satisfies*

(i) $y(1/2) = 1, y(1 - w) = \gamma - e^{-\alpha(1-w)}, \lim_{\delta \to 0} y(1) = \gamma - a^2$,

(ii) $1 + \delta < y(t) < \gamma$ *for* $t \in [1 - w, 1]$,

(iii) $y(t) = \gamma - e^{-\alpha t}$ *for* $t \in [0, 1/2 + \eta_3]$, *where* η_3 *is given by* $y(1/2 + \eta_3) = 1 + \delta$,

(iv) $y(t) = \gamma + (y(1) - \gamma) e^{-\alpha(t-1)}$ *for* $t \in [1, 3/2]$.

Proof. By $y(t) = \gamma - e^{-\alpha t}$ for $0 \leqslant t \leqslant 1 - w$ (see (2.3)) and $w \to 0$ as $\delta \to 0$ we obtain $y(1/2) = 1$, $y(1 - w) = \gamma - e^{-\alpha(1-w)} > 1$, and (iii) for δ sufficiently small. Further, (iv) follows from (2.3) by $a \leqslant y(s - 1) \leqslant 1$ for $1 \leqslant s \leqslant 3/2$. Using (2.8)–(2.10) and $1 + \delta < y(1 - w) < \gamma$ we find (ii) for δ small enough, and $\lim_{\delta \to 0} y(1) = \gamma - a^2$. Q.E.D.

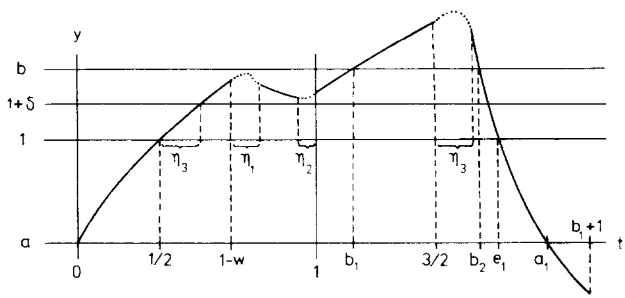

FIGURE 4

We now rely for the first time on conditions for b.

LEMMA 4.2. *Let* y *be the solution in Lemma* 4.1. *Then*

$$\max_{t \in [1/2, 1]} y(t) < b < \min_{t \in [3/2, 3/2 + \eta_3]} y(t) - \delta.$$

Proof. y is increasing in $[1/2, 1 - w]$, decreasing in $[1 - w + \eta_1, 1 - \eta_2]$. Moreover $(a - \delta) a^{-1} y(1 - w) \leqslant y(t) \leqslant y(1 - w) + \eta_1 c$ in $[1 - w, 1 - w + \eta_1]$ (see (3.5)) and, according to (3.10) and (3.11),

$$y(t) \leqslant y(1 - w + \eta_1) + \eta_2 c \qquad \text{for} \quad t \in [1 - \eta_2, 1].$$

Similarly

$$y(t) \geqslant y(3/2) \exp(-\alpha \eta_3) \qquad \text{for} \quad t \in [3/2, 3/2 + \eta_3].$$

Since $w, \eta_1, \eta_2, \eta_3 \to 0$, $y(1-w) \to \gamma - a^2$, and

$$y(3/2) \to \gamma + (\gamma - a^2 - \gamma) \cdot \exp(-\alpha/2) = \gamma - a^3 \quad \text{as} \quad \delta \to 0, \tag{4.4}$$

the lemma is obtained. Q.E.D.

In the interval $[1, 3/2]$ the solution y is strictly increasing according to

$$y(t) = \gamma + (y(1) - \gamma)\, e^{-\alpha(t-1)}, \qquad t \in [1, 3/2], \tag{4.5}$$

from a value below b to a value above b. Hence there is a unique $b_1 \in (1, 3/2)$ satisfying $y(b_1) = b$.

As $\delta \to 0$, $y(1) \to \gamma - a^2$. Therefore (4.5) implies

$$\lim_{\delta \to 0} b_1 = 5/4. \tag{4.6}$$

Since $1 + \delta \leqslant y(t) \leqslant b$ for $t \in [1/2 + \eta_3, b_1]$, the solution obeys

$$y(t) = y(\tfrac{3}{2} + \eta_3) \exp(-\alpha[t - \tfrac{3}{2} - \eta_3]) \quad \text{for} \quad t \in [\tfrac{3}{2} + \eta_3, b_1 + 1] \tag{4.7}$$

LEMMA 4.3. *The solution y determined in Lemma* 4.1 *satisfies*

$$y(b_1 + 1) < a - \delta.$$

Proof. We have

$$y(\tfrac{3}{2} + \eta_3) = y(\tfrac{3}{2})\, e^{-\alpha\eta_3} + \int_{3/2}^{3/2+\eta_3} e^{-\alpha(3/2+\eta_3-s)} f(\gamma - e^{-\alpha(s-1)})\, ds,$$

see Lemma 4.1(iii). By (4.5) we find $y(3/2 + \eta_3) \to \gamma - a^3$ as $\delta \to 0$. Now (4.7), (4.6) and $a \leqslant 1/2$ yield

$$\lim_{\delta \to 0} y(b_1 + 1) = (\gamma - a^3) \exp(-\alpha \cdot 3/4) = (1 + a - a^3)\, a^{3/2} < a. \text{Q.E.D.}$$

Because of (4.7) and the last two lemmas there are unique numbers $b_2 = b_2(w, \delta)$, $e_1 = e_1(w, \delta)$, $a_1 = a_1(w, \delta)$ in the interval $(3/2 + \eta_3, b_1 + 1)$ with

$$y(b_2) = b, \qquad y(e_1) = 1, \qquad y(a_1) = a, \qquad b_2 < e_1 < a_1.$$

Up to the time $t = b_1 + 1$ the solution y does not depend on the value of d. In the situations where the dependence on d requires attention we shall write y_d instead of y.

LEMMA 4.4. *For all $d \leqslant 0$*

$$y_d(t) \leqslant y_0(t) = y(b_1 + 1)\, e^{-\alpha(t - b_1 - 1)}, \qquad t \in [b_1 + 1, e_1 + 1 - \eta_6],$$

η_6 satisfying $1=(1+\delta)\exp(-\alpha\eta_6)$. Moreover $y_d(e_1+1-\eta_6)$ depends continuously on d.

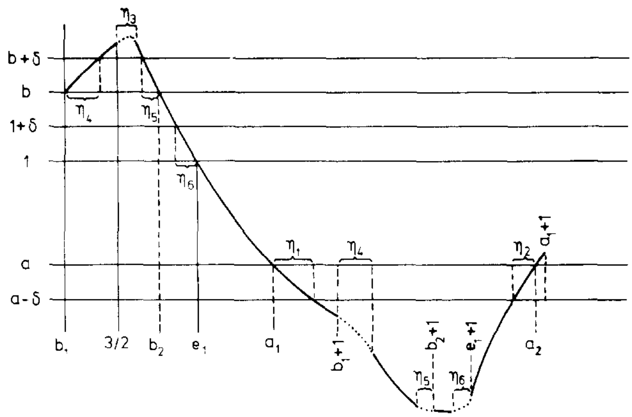

FIGURE 5

Proof of Lemma 4.4. Define η_4 by $b+\delta=\gamma+(b-\gamma)\exp(-\alpha\eta_4)$. For $t\in[b_1+1,\ b_1+1+\eta_4]$ the inequality of the lemma follows from $y_d(t)=y(b_1+1)\exp(-\alpha(t-b_1-1))+E_4(t)$, where

$$E_4(t)=\int_{b_1+1}^{t} e^{-\alpha(t-s)}f(\gamma+(b-\gamma)\,e^{-\alpha(s-b_1-1)})\,ds,$$

since $f(x)\leqslant 0$ for $x\geqslant b$. For the same t the equality follows from $f(x)=0$ if $x\geqslant b$ and $d=0$. Obviously $y_d(b_1+1+\eta_4)$ is continuous with respect to d. In a similar way, using (2.3) and (2.2), the lemma is proved for $t\in[b_1+1+\eta_4,\,b_2+1-\eta_5]$, η_5 defined by $b=(b+\delta)\exp(-\alpha\eta_5)$, then for $t\in[b_2+1-\eta_5,\,b_2+1]$, and finally for $t\in[b_2+1,\,e_1+1-\eta_6]$. Q.E.D.

LEMMA 4.5. *For all $d\leqslant 0$*

$$y_d(t)\leqslant y_0(t)=y_0(e_1+1-\eta_6)\,e^{-\alpha(t-e_1-1+\eta_6)}+E_6(t),$$
$$t\in[e_1+1-\eta_6,\,e_1+1],$$

where

$$E_6(t)=\int_{e_1+1-\eta_6}^{t} e^{-\alpha(t-s)}f((1+\delta)\,e^{-\alpha(s-e_1+\eta_6-1)})\,ds.$$

Moreover $y_d(e_1+1)$ depends continuously on d.

Proof. The assertions follow similar to the previous lemma from

$$y_d(t) = y_d(e_1 + 1 - \eta_6)\, e^{-\alpha(t - e_1 - 1 + \eta_6)} + E_6(t),$$
$$t \in [e_1 + 1 - \eta_6, e_1 + 1], \quad (4.8)$$

since η_6 and $E_6(t)$ do not depend on d. Q.E.D.

LEMMA 4.6.

$$y_d(t) = \gamma + (y_d(e_1 + 1) - \gamma)\, e^{-\alpha(t - e_1 - 1)} \leqslant y_0(t), \quad t \in [e_1 + 1, a_1 + 1], \quad (4.9)$$

$y_d(a_1 + 1)$ is continuous with respect to d.

LEMMA 4.7. *The solution y_0 satisfies*

$$y_0(t) < a - \delta \qquad \textit{for} \quad t \in [b_1 + 1, e_1 + 1], \qquad y_0(a_1 + 1) > a.$$

Proof. The first inequality results from Lemma 4.3, from $y_0(t) = y(t)$ for $t \leqslant b_1 + 1$, from Lemmas 4.4 and 4.5 and from the fact that $E_6(t) \leqslant c\eta_6$, $\eta_6 \to 0$ as $\delta \to 0$.

To establish the second inequality observe

$$\lim_{\delta \to 0} y_0(e_1 + 1) = \lim_{\delta \to 0} y_0(e_1 + 1 - \eta_6)$$

and that

$$y_0(t) = \exp(-\alpha(t - e_1)) \qquad \text{for} \quad t \in [e_1, e_1 + 1 - \eta_6], \qquad (4.10)$$

hence

$$\lim_{\delta \to 0} y_0(e_1 + 1) = \exp(-\alpha) = a^2.$$

This relation together with $y_0(a_1 + 1) = \gamma + (y_0(e_1 + 1) - \gamma)\, e^{-\alpha(a_1 - e_1)}$ (see (4.9)) implies

$$\lim_{\delta \to 0} y_0(a_1 + 1) = \gamma + (a^2 - \gamma)\, a = 1 + a^3 - a^2 > a. \qquad \text{Q.E.D.}$$

LEMMA 4.8.

$$\lim_{d \to -\infty} y_d(a_1 + 1) = -\infty. \qquad (4.11)$$

Proof. For $t \in [e_1 + 1, a_1 + 1]$, formula (4.9) holds. Since additionally a_1 and e_1 do not depend on d, relation (4.11) is valid if $y_d(e_1 + 1) \to -\infty$ as $d \to -\infty$. This relation holds if $y_d(e_1 + 1 - \eta_6) \to -\infty$ as $d \to -\infty$ (because of

(4.8); η_6 and E_6 are independent of d). Similarly the latter condition is satisfied if $y_d(b_2+1) \to -\infty$ as $d \to -\infty$. From the relations

$$y_d(b_2+1) \leqslant y_d(b_2+1-\eta_5)\, e^{-\alpha\eta_5}, \quad y_d(b_1+1+\eta_4) \leqslant y_0(b_1+1+\eta_4),$$

and

$$y_d(b_2+1-\eta_5) = \frac{d}{\alpha} + \left(y_d(b_1+1+\eta_4) - \frac{d}{\alpha}\right) \exp(-\alpha(b_2-b_1-\eta_5-\eta_4))$$

(see (2.3)), it follows that

$$y_d(b_2+1) \leqslant \frac{d}{\alpha}\,(1-e^{-\alpha(b_2-b_1-\eta_5-\eta_4)})\, e^{-\alpha\eta_5}$$
$$+ e^{-\alpha\eta_5} y_0(b_1+1+\eta_4)\, e^{-\alpha(b_2-b_1-\eta_5-\eta_4)},$$

hence $y_d(b_2+1)$ tends to $-\infty$ as $d \to -\infty$, completing the proof. Q.E.D.

As a conclusion from Lemmas 4.6–4.8 there is a number $d^* < 0$ such that

$$y_d(a_1+1) > a \qquad \text{for} \quad d \in (d^*, 0] \qquad \text{and} \qquad y_{d^*}(a_1+1) = a.$$

Since $y_d(e_1+1) \leqslant y_0(e_1+1) < a - \delta$, to each $d \in [d^*, 0]$ there corresponds a number $a_2 = a_2(d)$ such that

$$e_1 + 1 < a_2 \leqslant a_1 + 1, \qquad y_d(a_2) = a.$$

LEMMA 4.9. *For all $d \in [d^*, 0]$ the function $\psi_d\colon [-1, 0] \to \mathbb{R}$ defined by $\psi_d(t) = y_d(a_2(d) + t)$ is contained in D and*

$$V(\psi_d) = a_2(d) - a_1.$$

Proof. By Lemmas 4.3–4.7, $y_d(t) < a - \delta$ for $t \in (a_1 + \eta_1, e_1 + 1]$. Since $y_d(t) = y(t)$ for $t \leqslant b_1 + 1$, $a \leqslant y(d) \leqslant 1$ for $t \in [e_1, a_1]$ and y_d decays exponentially in $[a_1, a_1 + \eta_1]$. Because of Lemma 4.6, ψ_d satisfies Definition 1(ii). Since finally $a_2 - e_1 > 1$ and $V(\psi_d) = a_2 - a_1 \leqslant a_1 + 1 - a_1 = 1$, the function ψ_d has all properties to be contained in D. Q.E.D.

LEMMA 4.10.

$$v_1 < V(\psi_0) < v_p, \qquad V(\psi_{d^*}) = 1.$$

Proof. The equality holds since $a_2(d^*) = a_1 + 1$. To prove the inequality it is enough to show

$$\lim_{\delta \to 0} v_1 < \lim_{\delta \to 0} V(\psi_0) < \lim_{\delta \to 0} v_3 \tag{4.12}$$

(so that $F(V(\psi_0))$ is defined) and

$$F(V(\psi_0)) < V(\psi_0) \tag{4.13}$$

(so that $\partial F/\partial v > 1$ implies $V(\psi_0) < v_p$). We have

$$\lim_{\delta \to 0} V(\psi_0) = \frac{1}{\alpha} \log \frac{1 + a - a^2}{a} \tag{4.14}$$

because of $V(\psi_0) = a_2 - (e_1 + 1) + (e_1 + 1 - a_1)$,

$$a = y_0(a_1) = e^{-\alpha(a_1 - e_1)}, \qquad \lim_{\delta \to 0} y_0(e_1 + 1) = a^2$$

(see (4.10)) and

$$a = y_0(a_2) = \gamma + (y_0(e_1 + 1) - \gamma)\, e^{-\alpha(a_2 - (e_1 + 1))} \qquad \text{(see (4.9))}.$$

Also,

$$\lim_{\delta \to 0} v_3 = \frac{1}{\alpha} \log \frac{a^4 + a^3 + a + 1}{a^5 + a^4 + a^3 + a^2}.$$

This follows from $1 = F(v_3)$ which gives

$$e^{-\alpha(v_3 - \eta_1)} = e^{-\alpha} \frac{(\gamma - x(1))\, a}{x(1 - v_3 + \eta_1)},$$

from (4.14) and from (3.4), (3.8), (3.10), (3.19) applied to solutions $x = x_\varphi$ with $V(\varphi) = v_3$. Now $0 < a \leqslant 1/2$ and $\lim_{\delta \to 0} v_1 = 1/2$ yield (4.12).

To prove (4.13) let $\tilde{v} = V(\psi_0)$. Because of (3.27), $F(\tilde{v}) < \tilde{v}$ for small δ if $a(1 + a - x(1)) < x(1 - \tilde{v})$ for solutions $x = x_\varphi$ with $V(\varphi) = \tilde{v} \in I$. The last inequality is satisfied because (3.8), (3.4), (4.14), (3.10) and (3.19) imply

$$\lim_{\delta \to 0} x(1 - \tilde{v} + \eta_1) = 1 - a^2 + a^3$$

and

$$\lim_{\delta \to 0} x(1) = a(1 - a^2 + a^3)/(1 + a - a^2). \qquad \text{Q.E.D.}$$

THEOREM 3. *Let Condition 2 hold. Then for each number* $\alpha \geqslant \log 4$ *there is a bounded sequence* $(d_i(\alpha))_{i=1,2,\ldots}$ *of negtive numbers and an integer* $p(\alpha)$ *such that*

(i) *if* $d = d_i(\alpha)$ *then there is a periodic solution* $x = x_i$ *to Eq.* (2.1) *having* $p(\alpha) + i$ *maxima within one smallest period,*

(ii) *as* $i \to \infty$ *the period of* x_i *tends to* ∞.

Proof. Consider again the solution y_d to the initial condition $\varphi \in X$ with $V(\varphi) = w \in (F(v_1), F(v_1) + \delta)$. Let $v(k) = F^{-k}(w)$, $k \in \mathbb{N}$. Since $\partial F/\partial v > 1$, $v_1 < v(k) < v(k+1)$, $\lim_{k\to\infty} v(k) = v_p$.

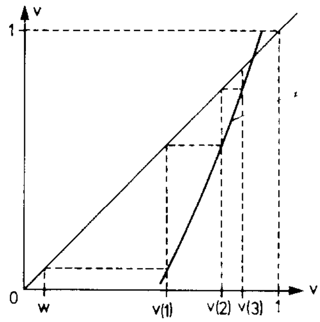

FIGURE 6

Lemma 4.10 implies there is $k_1 = k_1(\alpha)$ such that $v(k) \geqslant V(\psi_0)$ for all $k \geqslant k_1$.

Since $V(\psi_d) = a_2(d) - a_1$ depends continuously on $d \in [d^*, 0]$, Lemma 4.10 also implies that for each $k \geqslant k_1$ there is $d = d_i(\alpha)$, $i = k - k_1 + 1$, obeying $V(\psi_d) = v(k)$. Fix α, k and $d = d_i(\alpha)$. By $\psi_d \in D$ (Lemma 4.9) and $v(k) \in I, \ldots, v(1) \in I$, we obtain $V(T^k(\psi_d)) = F^k(v(k)) = w = V(\varphi)$. With $\varphi \in X$, it follows tat $T^k(\psi_d) = \varphi$. This implies that y_d is periodic with minimal period $a_{2(k+1)}$, where $t = a_j, j \geqslant 1$, denotes the jth time such that $y_d(t) = a$.

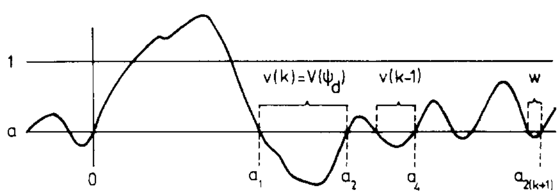

FIGURE 7

Since $a_{2(j+1)} - a_{2j} > 1$, $1 \leqslant j \leqslant k$, the period is larger than k which implies (ii). Between 0 and $a_1$there are two maxima, between a_2 and $a_{2(k+1)}$ there are k maxima, altogether $k + 2$. Therefore, (i) holds with $p(\alpha) = k_1 + 2$. Q.E.D.

5. Existence of a Solution Which Is Homoclinic to a Periodic Orbit

Throughout this section let again Condition 2 be valid and let y_d be the solution to (2.1) defined in Section 4 (for some fixed w as in Lemma 4.1).

Lemmas 4.9 and 4.10 show, since $a_2(d)$ depends continuously on d, that there is $\tilde{d} \in [d^*, 0]$ such that

$$V(\psi_{\tilde{d}}) = v_p. \tag{5.1}$$

The orbit of a function x solving (2.1) for all $t \in \mathbb{R}$ is defined as the set $\{x_t : t \in \mathbb{R}\} \subset C$, where $x_t(s) = x(t+s)$, $-1 \leqslant s \leqslant 0$. Equation (5.1) implies that $y_{\tilde{d},t}$ is on the orbit of the unstable periodic solution x_p for all $t \geqslant a_2(\tilde{d}) + 1$. The function F allows to extend $y_{\tilde{d}}$ to a solution of (2.1) for all $t \in \mathbb{R}$ in the following way:

Let $v_0 = w$, $v_k = F^{-k}(v_0)$, $k \in \mathbb{N}$. Obviously $v_k \to v_p$ as $k \to \infty$ (see Lemma 3.4 and (4.3)). Choose $\varphi_k \in X$ such that $V(\varphi_k) = v_k$. Let $x(k, t)$, $t > 0$, be the solution to φ_k and $t = a_{2k}$ the second positive time with $x(k, t) = a$. Let $t_0 = 0$, $t_{k+1} = t_k - a_{2k}$, $k \in \mathbb{N}_0$. Define $y_{\tilde{d}}(t_k + t) = x(k, t)$ for $k \in \mathbb{N}$, $t \in [0, a_{2k}]$, $t_k + t < -1$. Then by construction $y_{\tilde{d}}$ solves (2.1) for all $t \in \mathbb{R}$ as follows from Lemma 3.5 and Proposition 1.

Theorem 4. *On Condition 2 for each $\alpha \geqslant \log 4$ there is a number $d = d(\alpha)$ such that the solution $y_{d(\alpha)}$ is homoclinic to the periodic solution x_p.*

Proof. With $d(\alpha) = \tilde{d}$ as above we show that the orbit of $y_{\tilde{d}}$ belongs both to the stable and unstable manifold of the orbit of x_p (the unstable periodic solution determined in Section 3). The first assertion holds since $\tilde{y}_{d,t}$ is on the orbit of x_p for $t \geqslant a_2(\tilde{d}) + 1$. The second assertion follows from $y_{\tilde{d},t_k} \to x_{p,0}$ as $k \to \infty$. Q.E.D.

6. Existence of Chaos

In this section, $\delta > 0$, $d = \tilde{d}(\alpha)$ and $\tilde{w} = w = w(\delta) \in (F(v_1), F(v_1) + \delta) \subset \mathbb{R} \backslash I$ are held fixed. We extend the operator T, defined in Section 3, onto a one-dimensional neighborhood $N \subset X$ of the initial value $y_{\tilde{d},0}$ of the homoclinic solution.

Note that for $\varepsilon > 0$ sufficiently small, all solutions $y = y_w$ with initial value $\varphi \in X$, $w = V(\varphi) \in U := (\tilde{w} - \varepsilon, \tilde{w} + \varepsilon) \subset (F(v_1), F(v_1) + \delta)$ have all the properties described in Lemmas 4.1–4.3. Of course, the numbers b_1, b_2, e_1, a_1 depend continuously on $w \in U$. In addition the number $a_2 = a_2(w)$ is well defined as the second positive time such that $y_w(a_2) = a$, and is also continuous as a function of $w \in U$. The graph of y_w looks as in Figs. 4 and

5. Since $d<0$, the segment $y_{w,a_2} \in D$ but not $y_{w,a_2} \in X$. However, $V(y_{\tilde{w},a_2}) = v_p$ and the continuity of $a_2(w) - a_1(w)$ imply $a_2(w) - a_1(w) \in I$, $F(a_2(w) - a_1(w)) \in I$, $T(y_{w,a_2}) \in X$ for $w \in U$ provided ε is sufficiently small. For such $\varepsilon > 0$ we define

$$\begin{aligned} F(w) &:= F(a_2(w) - a_1(w)) \qquad \text{for all} \quad w \in U, \\ N := (V|_X)^{-1}(U), \quad T(\varphi) &:= T(y_{w,a_2}) \quad \text{for} \quad \varphi \in N \quad \text{with} \quad V(\varphi) = w. \end{aligned} \tag{6.1}$$

Lemma 6.1. *There is an open interval $U \subset [0,1]$ such that $\tilde{w} \in U$, $U \cap I = \varnothing$, $U \subset F(I)$ and such that* (6.1) *defines an extension of F onto $U \cup I$ satisfying*

(i) $F(\tilde{w}) = v_p$, $F(U) \subset I$,

(ii) *F is one-to-one on U,*

(iii) $V(T(\varphi)) = F(V(\varphi))$ *for all* $\varphi \in N$.

Proof. It remains to prove that for sufficiently small $\varepsilon > 0$ the function F is injective. Because of $\partial F/\partial v > 1$ on I, the latter assertion is true if $a_2(w) - a_1(w)$ is a strictly monotone function of w in a neighborhood of $\tilde{w}$. If $\varphi \in X$, $V(\varphi) = w \in (\tilde{w} - \varepsilon, \tilde{w} + \varepsilon)$, then Lemmas 4.1–4.6 and formulas (4.4)–(4.8) hold for the solution $y = y(\varphi) = y_w$ (note $d = \tilde{d}$ and y_d sometimes has to be replaced by y_w). Since $a_2 - a_1 = a_2 - (e_1 + 1) + 1 - (a_1 - e_1)$ and $a_1 - e_1$ is independent of w (see Fig. 5), consider $a_2 - (e_1 + 1)$, $e_1 = e_1(w)$. Equation (4.9) implies

$$a = \gamma + (y_w(e_1(w) + 1) - \gamma)\, e^{-\alpha(a_2(w) - e_1(w) - 1)}.$$

The proof is completed if $y_w(e_1(w) + 1)$ is strictly monotone. Indeed, by straightforward calculations it can be shown successively that $y_w(t)$ is strictly increasing with respect to w for $t = 1 - w$, $1 - w + \eta_1$, $1 - \eta_2$, 1, $3/2$, $3/2 + \eta_3$, $b_1(w) + 1$, $b_1(w) + 1 + \eta_4$, $b_2 + 1 - \eta_5$, $b_2 + 1$, $e_1 - \eta_6 + 1$, and finally for $t = e_1(w) + 1$. Q.E.D.

Theorem 5. *Let Condition* 2 *hold. Then to each $\alpha \geqslant \log 4$ there corresponds a number $d = d(\alpha)$ such that there is a one-dimensional set $X \subset C$ of initial conditions to Eq.* (2.1) *containing subsets $S \subset M \subset X$ with the following properties*:

(i) *To each solution x_φ, $\varphi \in M$, there corresponds a strictly increasing sequence $(a_i)_{i=1,2,\ldots}$ of positive numbers obeying $x_\varphi(a_i) = a$, $x_\varphi(t) \neq a$ for all positive $t \neq a_i$, $m_i(\varphi) = a_{2i} - a_{2i-1} < 1$,*

(ii) *there are infinitely many periodic solutions having pairwise* *different periods, in particular there is a number n_0 such that for each $n \in \mathbb{N}$, $n \geqslant n_0$, there is $\varphi \in M$ with x_φ periodic and*

$$\begin{aligned} m_n(\varphi) &= m_1(\varphi) \\ &\neq m_i(\varphi) \qquad \text{for} \quad i = 2, 3, \ldots, n-1, \end{aligned}$$

$a_{2n} - a_2$ being the period of x_φ,

(iii) *S is uncountable, does not contain initial values of periodic solutions, and*

$$\left.\begin{aligned} &\limsup_{n\to\infty} |m_i(\varphi) - m_i(\psi)| > 0 \\ &\liminf_{n\to\infty} |m_i(\varphi) - m_i(\psi)| = 0, \end{aligned}\right\} \quad \text{if} \quad \varphi, \psi \in S,\ \varphi \neq \psi,$$

$$\limsup_{n\to\infty} |m_i(\varphi) - m_i(\tilde{\psi})| > 0 \qquad \text{if} \quad \varphi \in S,\ \tilde{\psi} \in M \text{ and } x_{\tilde{\psi}} \text{ periodic.}$$

Proof. Theorem 1 applies to a restriction of the map F of Lemma 6.1. V being a homeomorphism of X onto $F(I)$ we obtain the theorem from $V \circ T = F \circ V$ (Lemma 6.1(iii)). M is the preimage of the maximal positively invariant set for the restriction of F. Note that some segments $x_{\varphi, a_{2i}}$, $\varphi \in M$, and distances $m_i(\varphi)$ are not given by iterates $T^j(\varphi)$ and $F^j(v)$, respectively. For example, if $\varphi \in N$ and $\varphi \neq y_{d,0}$ then $x_{\varphi, a_2} \in D \backslash X$ and $m_1(\varphi) \neq F(V(\varphi))$.
Q.E.D.

Note added in proof. An extension of this work is presented in U. an der Heiden and M. C. Mackey, The dynamics of production and destruction, *J. Math. Biol.*, in press.

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